Compactness and Incompactness on Singular Cardinals

Maxwell Levine

Universität Wien

Winter School in Abstract Analysis 2020
section Set Theory & Topology

January 30, 2020

Section 1

What Compactness Means for Singular Cardinals

Compactness Properties at Singular Cardinals

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ.

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ. The smallest such λ is the *cofinality* of κ and is denoted $\mathrm{cf}\,\kappa$.

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ. The smallest such λ is the *cofinality* of κ and is denoted $\mathrm{cf}\,\kappa$.

- $\aleph_0$, $\aleph_1$ are not singular, nor are inaccessible cardinals.

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ. The smallest such λ is the *cofinality* of κ and is denoted $\operatorname{cf} \kappa$.

- $\aleph_0$, $\aleph_1$ are not singular, nor are inaccessible cardinals.
- $\aleph_\omega$, $\aleph_{\omega^2}$, $\aleph_{\omega_1}$ are singular.

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ. The smallest such λ is the *cofinality* of κ and is denoted $\operatorname{cf} \kappa$.

- $\aleph_0$, $\aleph_1$ are not singular, nor are inaccessible cardinals.
- $\aleph_\omega$, $\aleph_{\omega^2}$, $\aleph_{\omega_1}$ are singular.
- When we talk about singulars, we are often interested in their successors, e.g. $\aleph_{\omega+1}$.

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ. The smallest such λ is the *cofinality* of κ and is denoted $\mathrm{cf}\,\kappa$.

- $\aleph_0$, $\aleph_1$ are not singular, nor are inaccessible cardinals.
- $\aleph_\omega$, $\aleph_{\omega^2}$, $\aleph_{\omega_1}$ are singular.
- When we talk about singulars, we are often interested in their successors, e.g. $\aleph_{\omega+1}$.
- A property exemplifies *compactness* if approximations of a possible object imply the existence of the object, e.g. the tree property.

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ. The smallest such λ is the *cofinality* of κ and is denoted $\mathrm{cf}\,\kappa$.

- $\aleph_0$, $\aleph_1$ are not singular, nor are inaccessible cardinals.
- $\aleph_\omega$, $\aleph_{\omega^2}$, $\aleph_{\omega_1}$ are singular.
- When we talk about singulars, we are often interested in their successors, e.g. $\aleph_{\omega+1}$.
- A property exemplifies *compactness* if approximations of a possible object imply the existence of the object, e.g. the tree property.
- Compact properties tend to follow from large cardinals, either directly or consistency-wise.

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ. The smallest such λ is the *cofinality* of κ and is denoted $\operatorname{cf} \kappa$.

- $\aleph_0$, $\aleph_1$ are not singular, nor are inaccessible cardinals.
- $\aleph_\omega$, $\aleph_{\omega^2}$, $\aleph_{\omega_1}$ are singular.
- When we talk about singulars, we are often interested in their successors, e.g. $\aleph_{\omega+1}$.
- A property exemplifies *compactness* if approximations of a possible object imply the existence of the object, e.g. the tree property.
- Compact properties tend to follow from large cardinals, either directly or consistency-wise.
- A property is non-compact if you have approximations of something that cannot exist, e.g. an Aronszajn tree or $\square_\kappa$.

Compactness Properties at Singular Cardinals

Definition

A cardinal κ is *singular* if there is some $\lambda < \kappa$ and some function $f : \lambda \to \kappa$ unbounded in κ. The smallest such λ is the *cofinality* of κ and is denoted $\operatorname{cf} \kappa$.

- $\aleph_0$, $\aleph_1$ are not singular, nor are inaccessible cardinals.
- $\aleph_\omega$, $\aleph_{\omega^2}$, $\aleph_{\omega_1}$ are singular.
- When we talk about singulars, we are often interested in their successors, e.g. $\aleph_{\omega+1}$.
- A property exemplifies *compactness* if approximations of a possible object imply the existence of the object, e.g. the tree property.
- Compact properties tend to follow from large cardinals, either directly or consistency-wise.
- A property is non-compact if you have approximations of something that cannot exist, e.g. an Aronszajn tree or $\square_\kappa$.
- Non-compactness is a key property of L and other inner models.

Compact *Behavior* of Singular Cardinals

Compact *Behavior* of Singular Cardinals

Theorem (Easton)

The continuum function $\kappa \mapsto 2^\kappa$ on **regular** *cardinals is constrained only by:*

- $\lambda \leq \kappa \implies 2^\lambda \leq 2^\kappa$ *(monotonicity)*
- $\text{cf}(2^\kappa) > \kappa$ *(König's Theorem).*

Compact *Behavior* of Singular Cardinals

Theorem (Easton)

The continuum function $\kappa \mapsto 2^\kappa$ *on* **regular** *cardinals is constrained only by:*

- $\lambda \leq \kappa \implies 2^\lambda \leq 2^\kappa$ *(monotonicity)*
- $\mathrm{cf}(2^\kappa) > \kappa$ *(König's Theorem).*

Theorem (Silver)

GCH *cannot fail for the first time at a singular of uncountable cofinality.*

Compact *Behavior* of Singular Cardinals

Theorem (Easton)

The continuum function $\kappa \mapsto 2^\kappa$ *on* **regular** *cardinals is constrained only by:*

- $\lambda \leq \kappa \implies 2^\lambda \leq 2^\kappa$ *(monotonicity)*
- $\mathrm{cf}(2^\kappa) > \kappa$ *(König's Theorem).*

Theorem (Silver)

GCH *cannot fail for the first time at a singular of uncountable cofinality.*

Theorem (Magidor)

Relative to a supercompact cardinal, it is consistent that GCH *holds below* $\aleph_\omega$ *but* $2^{\aleph_\omega} > \aleph_{\omega+1}$.

Compact *Behavior* of Singular Cardinals

Theorem (Easton)

The continuum function $\kappa \mapsto 2^\kappa$ *on* **regular** *cardinals is constrained only by:*

- $\lambda \leq \kappa \implies 2^\lambda \leq 2^\kappa$ *(monotonicity)*
- $\mathrm{cf}(2^\kappa) > \kappa$ *(König's Theorem).*

Theorem (Silver)

GCH *cannot fail for the first time at a singular of uncountable cofinality.*

Theorem (Magidor)

Relative to a supercompact cardinal, it is consistent that GCH *holds below* $\aleph_\omega$ *but* $2^{\aleph_\omega} > \aleph_{\omega+1}$.

Theorem (Shelah)

If $\aleph_\omega$ *is a strong limit then* $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Section 2

The Non-Compactness of Stationary Reflection (joint w/ Sy-David Friedman)

Defining Stationary Reflection

Defining Stationary Reflection

Definition

If κ is a cardinal and $S \subseteq \kappa$ a stationary set, then S *reflects at* α if $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary. S *reflects* if it reflects at some $\alpha < \kappa$.

Defining Stationary Reflection

Definition

If κ is a cardinal and $S \subseteq \kappa$ a stationary set, then S *reflects at* α if $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary. S *reflects* if it reflects at some $\alpha < \kappa$.

Definition

The *stationary reflection property* $\mathsf{SR}(S)$ holds if every stationary subset of S reflects.

Defining Stationary Reflection

Definition

If κ is a cardinal and $S \subseteq \kappa$ a stationary set, then S *reflects at* α if $\mathrm{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary. S *reflects* if it reflects at some $\alpha < \kappa$.

Definition

The *stationary reflection property* $\mathsf{SR}(S)$ holds if every stationary subset of S reflects.

Examples

Jensen: In L, $\mathsf{SR}(\kappa)$ holds if any only if κ is weakly compact.

Harr.-Sh.: $\mathrm{Con}(\mathsf{SR}(\aleph_2 \cap \mathrm{cof}(\omega))) \iff \mathrm{Con}(\exists$ a Mahlo cardinal$)$.

Sargsyan: $\mathrm{Con}(\aleph_\omega$ a strong limit $\wedge\, \mathsf{SR}(\aleph_{\omega+1})) \implies$ $\mathrm{Con}(\exists$ a Woodin cardinal with a strong cardinal below it$)$.

Solovay?: κ supercompact $\implies \mathsf{SR}(\lambda \cap \mathrm{cof}(< \kappa))$ for all regular $\lambda \geq \kappa$.

"Eastonizing" Stationary Reflection for Fixed Cofinality

"Eastonizing" Stationary Reflection for Fixed Cofinality

Proposition

- $\mathrm{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds vacuously if* $\kappa \leq \aleph_n$*;*

"Eastonizing" Stationary Reflection for Fixed Cofinality

Proposition

- $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds vacuously if* $\kappa \leq \aleph_n$*;*
- $\mathsf{SR}(\aleph_{n+1} \cap \mathrm{cof}(\aleph_n))$ *fails;*

"Eastonizing" Stationary Reflection for Fixed Cofinality

Proposition

- $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds vacuously if* $\kappa \leq \aleph_n$;
- $\mathsf{SR}(\aleph_{n+1} \cap \mathrm{cof}(\aleph_n))$ *fails;*
- $\mathsf{SR}(\lambda \cap \mathrm{cof}(\aleph_n)) \iff \mathsf{SR}(\mathrm{cf}\,\lambda \cap \mathrm{cof}(\aleph_n))$.

"Eastonizing" Stationary Reflection for Fixed Cofinality

Proposition

- $\mathrm{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds vacuously if* $\kappa \leq \aleph_n$*;*
- $\mathrm{SR}(\aleph_{n+1} \cap \mathrm{cof}(\aleph_n))$ *fails;*
- $\mathrm{SR}(\lambda \cap \mathrm{cof}(\aleph_n)) \iff \mathrm{SR}(\mathrm{cf}\,\lambda \cap \mathrm{cof}(\aleph_n))$.

Theorem (Friedman-L.)

Suppose χ *is a supercompact cardinal in* V *such that* GCH *holds above* χ*. Let* F *be a* 2*-valued function on the class of regular cardinals* $\geq \chi$*. Then there is a forcing extension* $W \supset V$ *in which:*

"Eastonizing" Stationary Reflection for Fixed Cofinality

Proposition

- $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds vacuously if* $\kappa \leq \aleph_n$;
- $\mathsf{SR}(\aleph_{n+1} \cap \mathrm{cof}(\aleph_n))$ *fails;*
- $\mathsf{SR}(\lambda \cap \mathrm{cof}(\aleph_n)) \iff \mathsf{SR}(\mathrm{cf}\,\lambda \cap \mathrm{cof}(\aleph_n))$.

Theorem (Friedman-L.)

Suppose χ is a supercompact cardinal in V such that GCH *holds above χ. Let F be a 2-valued function on the class of regular cardinals $\geq \chi$. Then there is a forcing extension $W \supset V$ in which:*

- $\chi = \aleph_{n+2}$;

"Eastonizing" Stationary Reflection for Fixed Cofinality

Proposition

- $\mathrm{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds vacuously if* $\kappa \leq \aleph_n$;
- $\mathrm{SR}(\aleph_{n+1} \cap \mathrm{cof}(\aleph_n))$ *fails;*
- $\mathrm{SR}(\lambda \cap \mathrm{cof}(\aleph_n)) \iff \mathrm{SR}(\mathrm{cf}\,\lambda \cap \mathrm{cof}(\aleph_n))$.

Theorem (Friedman-L.)

Suppose χ *is a supercompact cardinal in* V *such that* GCH *holds above* χ. *Let* F *be a* 2*-valued function on the class of regular cardinals* $\geq \chi$. *Then there is a forcing extension* $W \supset V$ *in which:*

- $\chi = \aleph_{n+2}$;
- *cofinalities and* GCH *are preserved above* χ;

“Eastonizing” Stationary Reflection for Fixed Cofinality

Proposition

- $\mathrm{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds vacuously if* $\kappa \leq \aleph_n$;
- $\mathrm{SR}(\aleph_{n+1} \cap \mathrm{cof}(\aleph_n))$ *fails;*
- $\mathrm{SR}(\lambda \cap \mathrm{cof}(\aleph_n)) \iff \mathrm{SR}(\mathrm{cf}\,\lambda \cap \mathrm{cof}(\aleph_n))$.

Theorem (Friedman-L.)

Suppose χ *is a supercompact cardinal in* V *such that* GCH *holds above* χ. *Let* F *be a* 2*-valued function on the class of regular cardinals* $\geq \chi$. *Then there is a forcing extension* $W \supset V$ *in which:*

- $\chi = \aleph_{n+2}$;
- *cofinalities and* GCH *are preserved above* χ;
- *for all regular* $\kappa \in W$ *such that* $\kappa \geq \aleph_{n+2}$, *there is a non-reflecting stationary subset of* $\kappa \cap \mathrm{cof}(\aleph_n)$ *if and only if* $F(\kappa) = 1$.

A Snappier Corollary

A Snappier Corollary

Remark

The *successors* of singulars present the most difficult cases.

A Snappier Corollary

Remark

The *successors* of singulars present the most difficult cases.

Corollary (Friedman-L.)

Relative to the consistency of a supercompact cardinal, it is consistent that there is a model in which for all regular $\kappa \geq \aleph_{n+2}$, $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds if and only if* $\varphi(\kappa)$ *holds where* $\varphi(\kappa)$ *could be any of the following:*

A Snappier Corollary

Remark

The *successors* of singulars present the most difficult cases.

Corollary (Friedman-L.)

Relative to the consistency of a supercompact cardinal, it is consistent that there is a model in which for all regular $\kappa \geq \aleph_{n+2}$, $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_n))$ *holds if and only if* $\varphi(\kappa)$ *holds where* $\varphi(\kappa)$ *could be any of the following:*

- κ *is the successor of a regular cardinal;*
- κ *is the successor of a singular cardinal;*
- κ *is inaccessible;*
- κ *is not inaccessible;*
- $\kappa = \aleph_{n+17}$*;*
- $\kappa \neq \aleph_{n+17}$*.*

Further Directions for Stationary Reflection

Further Directions for Stationary Reflection

Question

Can we get the same result for $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_{\omega+1}))$*?*

Further Directions for Stationary Reflection

Question

Can we get the same result for $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_{\omega+1}))$*?*

Conjecture

Probably, but then we need to force approachability "manually" and then the forcing will be even more complicated.

Further Directions for Stationary Reflection

Question

Can we get the same result for $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_{\omega+1}))$*?*

Conjecture

Probably, but then we need to force approachability "manually" and then the forcing will be even more complicated.

Question

Suppose that F *is a function on a class of regular cardinals* $\geq \aleph_2$ *to itself such that* $F(\kappa)^+ < \kappa$ *for all* $\kappa \in \mathrm{dom}(F)$*. Is it possible to obtain a model such that* $\mathsf{SR}(\kappa \cap \mathrm{cof}(\theta))$ *holds precisely when* $F(\kappa) = \theta$*?*

Further Directions for Stationary Reflection

Question

Can we get the same result for $\mathsf{SR}(\kappa \cap \mathrm{cof}(\aleph_{\omega+1}))$*?*

Conjecture

Probably, but then we need to force approachability "manually" and then the forcing will be even more complicated.

Question

Suppose that F *is a function on a class of regular cardinals* $\geq \aleph_2$ *to itself such that* $F(\kappa)^+ < \kappa$ *for all* $\kappa \in \mathrm{dom}(F)$*. Is it possible to obtain a model such that* $\mathsf{SR}(\kappa \cap \mathrm{cof}(\theta))$ *holds precisely when* $F(\kappa) = \theta$*?*

Conjecture

Absolutely not.

Section 3

The Situation with Square (joint w/ Sy-David Friedman and Dima Sinapova)

$\square_\kappa$ and $\square_\kappa^*$

$\square_\kappa$ and $\square^*_\kappa$

Definition

We say that $\square_\kappa$ holds if there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

$\square_\kappa$ and $\square^*_\kappa$

Definition

We say that $\square_\kappa$ holds if there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa^+) \rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

- C_α is a club in α;

$\Box_\kappa$ and $\Box^*_\kappa$

Definition

We say that $\Box_\kappa$ holds if there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

- C_α is a club in α;
- C_α has order-type $\leq \kappa$;

$\square_\kappa$ and $\square^*_\kappa$

Definition

We say that $\square_\kappa$ holds if there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

- C_α is a club in α;
- C_α has order-type $\leq \kappa$;
- $\forall \beta \in \lim C_\alpha,\ C_\alpha \cap \beta = C_\beta$.

$\square_\kappa$ and $\square^*_\kappa$

Definition

We say that $\square_\kappa$ holds if there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

- C_α is a club in α;
- C_α has order-type $\leq \kappa$;
- $\forall \beta \in \lim C_\alpha,\ C_\alpha \cap \beta = C_\beta$.

Fact

Given a $\square_\kappa$-sequence, there is no club $D \subseteq \kappa^+$ such that $\forall \alpha \in \lim D,\ D \cap \alpha = C_\alpha$. In other words, the sequence has no thread.

$\square_\kappa$ and $\square^*_\kappa$

Definition

We say that $\square_\kappa$ holds if there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

- C_α is a club in α;
- C_α has order-type $\leq \kappa$;
- $\forall \beta \in \lim C_\alpha,\ C_\alpha \cap \beta = C_\beta$.

Fact

Given a $\square_\kappa$-sequence, there is no club $D \subseteq \kappa^+$ such that $\forall \alpha \in \lim D,\ D \cap \alpha = C_\alpha$. In other words, the sequence has no thread.

Definition

We say that $\square^*_\kappa$ ("weak square") holds if there is a sequence $\langle \mathcal{C}_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

$\square_\kappa$ and $\square^*_\kappa$

Definition

We say that $\square_\kappa$ holds if there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

- C_α is a club in α;
- C_α has order-type $\leq \kappa$;
- $\forall \beta \in \lim C_\alpha,\ C_\alpha \cap \beta = C_\beta$.

Fact

Given a $\square_\kappa$-sequence, there is no club $D \subseteq \kappa^+$ such that $\forall \alpha \in \lim D,\ D \cap \alpha = C_\alpha$. In other words, the sequence has no thread.

Definition

We say that $\square^*_\kappa$ ("weak square") holds if there is a sequence $\langle \mathcal{C}_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for all $\alpha \in \lim(\kappa^+)$:

- $|\mathcal{C}_\alpha| \leq \kappa$;
- for all $C \in \mathcal{C}_\alpha$, C is a club in α of order-type $\leq \kappa$;
- for all $C \in \mathcal{C}_\alpha, \beta \in \lim C,\ C \cap \beta \in \mathcal{C}_\beta$.

Scales

Scales

Definition

If $f, g : \tau \mapsto \mathrm{ON}$, then $f <^* g$ if there is a $j < \tau$ such that $i \geq j \implies f(i) < g(i)$.

Scales

Definition

If $f, g : \tau \mapsto \mathrm{ON}$, then $f <^* g$ if there is a $j < \tau$ such that $i \geq j \implies f(i) < g(i)$.

If λ is a singular cardinal and $\langle \lambda_i : i < \operatorname{cf} \lambda \rangle$ is a sequence of regular cardinals converging to λ, a *scale* on λ is a sequence of functions $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:

Scales

Definition

If $f, g : \tau \mapsto \text{ON}$, then $f <^* g$ if there is a $j < \tau$ such that $i \geq j \implies f(i) < g(i)$.

If λ is a singular cardinal and $\langle \lambda_i : i < \text{cf}\,\lambda \rangle$ is a sequence of regular cardinals converging to λ, a *scale* on λ is a sequence of functions $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:

- The f_α's are in $\prod_{i < \text{cf}\,\lambda} \lambda_i$;

Scales

Definition

If $f, g : \tau \mapsto \mathrm{ON}$, then $f <^* g$ if there is a $j < \tau$ such that $i \geq j \implies f(i) < g(i)$.

If λ is a singular cardinal and $\langle \lambda_i : i < \mathrm{cf}\,\lambda \rangle$ is a sequence of regular cardinals converging to λ, a *scale* on λ is a sequence of functions $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:

- The f_α's are in $\prod_{i < \mathrm{cf}\,\lambda} \lambda_i$;
- The f_α's are $<^*$-increasing;

Scales

Definition

If $f, g : \tau \mapsto \mathrm{ON}$, then $f <^* g$ if there is a $j < \tau$ such that $i \geq j \implies f(i) < g(i)$.

If λ is a singular cardinal and $\langle \lambda_i : i < \mathrm{cf}\,\lambda \rangle$ is a sequence of regular cardinals converging to λ, a *scale* on λ is a sequence of functions $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:

- The f_α's are in $\prod_{i<\mathrm{cf}\,\lambda} \lambda_i$;
- The f_α's are $<^*$-increasing;
- The sequence is cofinal in the product $\prod_{i<\mathrm{cf}\,\lambda} \lambda_i$ with respect to $<^*$.

Scales

Definition

If $f, g : \tau \mapsto \mathrm{ON}$, then $f <^* g$ if there is a $j < \tau$ such that $i \geq j \implies f(i) < g(i)$.

If λ is a singular cardinal and $\langle \lambda_i : i < \operatorname{cf} \lambda \rangle$ is a sequence of regular cardinals converging to λ, a *scale* on λ is a sequence of functions $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:

- The f_α's are in $\prod_{i<\operatorname{cf}\lambda} \lambda_i$;
- The f_α's are $<^*$-increasing;
- The sequence is cofinal in the product $\prod_{i<\operatorname{cf}\lambda} \lambda_i$ with respect to $<^*$.

Theorem (Shelah)

If λ is a singular cardinal then there is a product of regular cardinals $\prod_{i<\operatorname{cf}\lambda} \lambda_i$ with $\sup_{i<\operatorname{cf}\lambda} \lambda_i = \lambda$ that carries a scale.

Connections Between Squares and Scales

Connections Between Squares and Scales

Definition

Let $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ be a scale on a product $\langle \lambda_i : i < \operatorname{cf} \lambda \rangle$.

Connections Between Squares and Scales

Definition

Let $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ be a scale on a product $\langle \lambda_i : i < \operatorname{cf} \lambda \rangle$.

- A function h is an *exact upper bound* of $\langle f_\beta : \beta < \alpha \rangle$ if $g <^* h$ implies that there is some $\beta < \alpha$ such that $g <^* f_\beta$.

Connections Between Squares and Scales

Definition

Let $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ be a scale on a product $\langle \lambda_i : i < \operatorname{cf} \lambda \rangle$.

- A function h is an *exact upper bound* of $\langle f_\beta : \beta < \alpha \rangle$ if $g <^* h$ implies that there is some $\beta < \alpha$ such that $g <^* f_\beta$.
- We say that α is a *good point* if $\langle f_\beta : \beta < \alpha \rangle$ has an exact upper bound h such that $\operatorname{cf}(h(i)) = \operatorname{cf}(\alpha)$ for large i.

Connections Between Squares and Scales

Definition

Let $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ be a scale on a product $\langle \lambda_i : i < \operatorname{cf} \lambda \rangle$.

- A function h is an *exact upper bound* of $\langle f_\beta : \beta < \alpha \rangle$ if $g <^* h$ implies that there is some $\beta < \alpha$ such that $g <^* f_\beta$.
- We say that α is a *good point* if $\langle f_\beta : \beta < \alpha \rangle$ has an exact upper bound h such that $\operatorname{cf}(h(i)) = \operatorname{cf}(\alpha)$ for large i.
- We say that $\vec{f}$ is *good scale* if there are club-many good points in λ^+ of cofinality $> \operatorname{cf} \lambda$.

Connections Between Squares and Scales

Definition

Let $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ be a scale on a product $\langle \lambda_i : i < \operatorname{cf} \lambda \rangle$.

- A function h is an *exact upper bound* of $\langle f_\beta : \beta < \alpha \rangle$ if $g <^* h$ implies that there is some $\beta < \alpha$ such that $g <^* f_\beta$.
- We say that α is a *good point* if $\langle f_\beta : \beta < \alpha \rangle$ has an exact upper bound h such that $\operatorname{cf}(h(i)) = \operatorname{cf}(\alpha)$ for large i.
- We say that $\vec{f}$ is *good scale* if there are club-many good points in λ^+ of cofinality $> \operatorname{cf} \lambda$.

Fact

If λ is singular, then $\square^*_\lambda$ implies that all scales on λ are good.

Connections Between Squares and Scales

Definition

Let $\vec{f} = \langle f_\alpha : \alpha < \lambda^+ \rangle$ be a scale on a product $\langle \lambda_i : i < \operatorname{cf} \lambda \rangle$.

- A function h is an *exact upper bound* of $\langle f_\beta : \beta < \alpha \rangle$ if $g <^* h$ implies that there is some $\beta < \alpha$ such that $g <^* f_\beta$.
- We say that α is a *good point* if $\langle f_\beta : \beta < \alpha \rangle$ has an exact upper bound h such that $\operatorname{cf}(h(i)) = \operatorname{cf}(\alpha)$ for large i.
- We say that $\vec{f}$ is *good scale* if there are club-many good points in λ^+ of cofinality $> \operatorname{cf} \lambda$.

Fact

If λ is singular, then $\square^*_\lambda$ implies that all scales on λ are good.

Fact

If κ is supercompact and $\operatorname{cf} \lambda < \kappa < \lambda$, then every scale on λ is bad.

Non-Compactness of $\square_\kappa$ at $\aleph_\omega$

Non-Compactness of $\square_\kappa$ at $\aleph_\omega$

Theorem (Cummings, Foreman, and Magidor / Krueger)

Assuming the consistency of a supercompact cardinal, there is a model where $\square_{\aleph_n}$ holds for all $n < \omega$ and:

Non-Compactness of $\square_\kappa$ at $\aleph_\omega$

Theorem (Cummings, Foreman, and Magidor / Krueger)

Assuming the consistency of a supercompact cardinal, there is a model where $\square_{\aleph_n}$ holds for all $n < \omega$ and:

CFM $\square_{\aleph_\omega}$ *fails.*

Non-Compactness of $\square_\kappa$ at $\aleph_\omega$

Theorem (Cummings, Foreman, and Magidor / Krueger)

Assuming the consistency of a supercompact cardinal, there is a model where $\square_{\aleph_n}$ holds for all $n < \omega$ and:

CFM $\square_{\aleph_\omega}$ *fails.*

K $\square^*_{\aleph_\omega}$ *fails, i.e. there is no $\aleph_{\omega+1}$-special Aronszajn tree. In fact, all scales on $\aleph_\omega$ are bad.*

A Weak Non-Compactness Result

A Weak Non-Compactness Result

Theorem (Friedman, L., Sinapova)

Assuming the consistency of a supercompact cardinal, there is a model in which:

- λ *is singular of uncountable cofinality;*

A Weak Non-Compactness Result

Theorem (Friedman, L., Sinapova)

Assuming the consistency of a supercompact cardinal, there is a model in which:

- λ *is singular of uncountable cofinality;*
- *there is a nonstationary set* $A \subset \lambda$ *such that* $\square_\delta$ *holds for all* $\delta \in A$*;*

A Weak Non-Compactness Result

Theorem (Friedman, L., Sinapova)

Assuming the consistency of a supercompact cardinal, there is a model in which:

- λ *is singular of uncountable cofinality;*
- *there is a nonstationary set* $A \subset \lambda$ *such that* $\square_\delta$ *holds for all* $\delta \in A$*;*
- $\square_\lambda$ *fails;*

A Weak Non-Compactness Result

Theorem (Friedman, L., Sinapova)

Assuming the consistency of a supercompact cardinal, there is a model in which:

- λ *is singular of uncountable cofinality;*
- *there is a nonstationary set* $A \subset \lambda$ *such that* $\square_\delta$ *holds for all* $\delta \in A$*;*
- $\square_\lambda$ *fails;*
- *more precisely, there is a bad scale on* λ*.*

Proof of the Weak Non-Compactness Result

Proof of the Weak Non-Compactness Result

- Start with a κ that is indestructibly supercompact under κ-directed-closed forcings.

Proof of the Weak Non-Compactness Result

- Start with a κ that is indestructibly supercompact under κ-directed-closed forcings.
- There is a standard forcing $\mathbb{S}_\delta$ for adding a $\square_\delta$-sequence. Force with a product $\mathbb{S} := \prod_{\delta \in A} \mathbb{S}_\delta$ where $A \subset \lambda \setminus \kappa$ is non-stationary. $\mathbb{S}$ is cofinality-preserving.

Proof of the Weak Non-Compactness Result

- Start with a κ that is indestructibly supercompact under κ-directed-closed forcings.
- There is a standard forcing $\mathbb{S}_\delta$ for adding a $\square_\delta$-sequence. Force with a product $\mathbb{S} := \prod_{\delta \in A} \mathbb{S}_\delta$ where $A \subset \lambda \setminus \kappa$ is non-stationary. $\mathbb{S}$ is cofinality-preserving.
- There is also a standard forcing $\mathbb{T}_{\delta,\ell}$ with $\ell \leq \delta$ regular defined in an extension $V[\mathbb{S}_\delta]$ such that $\mathbb{S}_\delta * \mathbb{T}_{\delta,\ell}$ is ℓ-directed closed.

Proof of the Weak Non-Compactness Result

- Start with a κ that is indestructibly supercompact under κ-directed-closed forcings.
- There is a standard forcing $\mathbb{S}_\delta$ for adding a $\square_\delta$-sequence. Force with a product $\mathbb{S} := \prod_{\delta \in A} \mathbb{S}_\delta$ where $A \subset \lambda \setminus \kappa$ is non-stationary. $\mathbb{S}$ is cofinality-preserving.
- There is also a standard forcing $\mathbb{T}_{\delta,\ell}$ with $\ell \leq \delta$ regular defined in an extension $V[\mathbb{S}_\delta]$ such that $\mathbb{S}_\delta * \mathbb{T}_{\delta,\ell}$ is ℓ-directed closed.
- A is avoided by a club $\langle \lambda_\xi : \xi < \operatorname{cf} \lambda \rangle$. There is a function F with domain A such that for all $\delta \in A$, $F(\delta)$ is a regular cardinal $\leq \delta$ and such that $F(\delta) \geq \lambda_\xi^+$ if $\delta \in A \setminus \lambda_\xi^+$.

Proof of the Weak Non-Compactness Result

- Start with a κ that is indestructibly supercompact under κ-directed-closed forcings.
- There is a standard forcing $\mathbb{S}_\delta$ for adding a $\square_\delta$-sequence. Force with a product $\mathbb{S} := \prod_{\delta \in A} \mathbb{S}_\delta$ where $A \subset \lambda \setminus \kappa$ is non-stationary. $\mathbb{S}$ is cofinality-preserving.
- There is also a standard forcing $\mathbb{T}_{\delta,\ell}$ with $\ell \leq \delta$ regular defined in an extension $V[\mathbb{S}_\delta]$ such that $\mathbb{S}_\delta * \mathbb{T}_{\delta,\ell}$ is ℓ-directed closed.
- A is avoided by a club $\langle \lambda_\xi : \xi < \mathrm{cf}\,\lambda \rangle$. There is a function F with domain A such that for all $\delta \in A$, $F(\delta)$ is a regular cardinal $\leq \delta$ and such that $F(\delta) \geq \lambda_\xi^+$ if $\delta \in A \setminus \lambda_\xi^+$.
- Use this idea to (carefully) embed $\mathbb{S}$ into a forcing $\mathbb{Q}$ which is κ-directed closed and which factors as product of a λ_ξ-sized forcing with a λ_ξ^+-distributive forcing for all $\xi < \mathrm{cf}\,\lambda$, which implies that it preserves λ^+.

Proof of the Weak Non-Compactness Result

- Start with a κ that is indestructibly supercompact under κ-directed-closed forcings.
- There is a standard forcing $\mathbb{S}_\delta$ for adding a $\square_\delta$-sequence. Force with a product $\mathbb{S} := \prod_{\delta \in A} \mathbb{S}_\delta$ where $A \subset \lambda \setminus \kappa$ is non-stationary. $\mathbb{S}$ is cofinality-preserving.
- There is also a standard forcing $\mathbb{T}_{\delta,\ell}$ with $\ell \leq \delta$ regular defined in an extension $V[\mathbb{S}_\delta]$ such that $\mathbb{S}_\delta * \mathbb{T}_{\delta,\ell}$ is ℓ-directed closed.
- A is avoided by a club $\langle \lambda_\xi : \xi < \operatorname{cf} \lambda \rangle$. There is a function F with domain A such that for all $\delta \in A$, $F(\delta)$ is a regular cardinal $\leq \delta$ and such that $F(\delta) \geq \lambda_\xi^+$ if $\delta \in A \setminus \lambda_\xi^+$.
- Use this idea to (carefully) embed $\mathbb{S}$ into a forcing $\mathbb{Q}$ which is κ-directed closed and which factors as product of a λ_ξ-sized forcing with a λ_ξ^+-distributive forcing for all $\xi < \operatorname{cf} \lambda$, which implies that it preserves λ^+.
- κ remains supercompact in $V[\mathbb{Q}]$, which means that every scale on λ^+ is bad. Because $\mathbb{Q}/\mathbb{S}$ preserves λ^+, this implies that there must have been a bad scale in $V[\mathbb{S}]$, hence $\square_\lambda$ fails in $V[\mathbb{S}]$.

Weak Silver's Theorem for Square

Theorem (Friedman, L., Sinapova)

Suppose:

- λ *is a singular strong limit cardinal of uncountable cofinality* μ*;*

Weak Silver's Theorem for Square

Theorem (Friedman, L., Sinapova)

Suppose:

- λ *is a singular strong limit cardinal of uncountable cofinality* μ*;*
- $\square_\delta$ *holds for stationarily-many* $\delta < \lambda$*;*

Weak Silver's Theorem for Square

Theorem (Friedman, L., Sinapova)

Suppose:

- λ *is a singular strong limit cardinal of uncountable cofinality* μ*;*
- $\square_\delta$ *holds for stationarily-many* $\delta < \lambda$*;*
- *every scale on* λ *is good.*

Weak Silver's Theorem for Square

Theorem (Friedman, L., Sinapova)

Suppose:

- λ *is a singular strong limit cardinal of uncountable cofinality* μ*;*
- $\Box_\delta$ *holds for stationarily-many* $\delta < \lambda$*;*
- *every scale on* λ *is good.*

Then there is a partial $\Box_\lambda$*-sequence on* $\lambda^+ \cap \mathrm{cof}(> \mu)$*,*

Weak Silver's Theorem for Square

Theorem (Friedman, L., Sinapova)

Suppose:

- λ *is a singular strong limit cardinal of uncountable cofinality* μ*;*
- $\square_\delta$ *holds for stationarily-many* $\delta < \lambda$*;*
- *every scale on* λ *is good.*

Then there is a partial $\square_\lambda$*-sequence on* $\lambda^+ \cap \mathrm{cof}(> \mu)$*,*
i.e. there is some $Z \supset \lim(\lambda^+) \cap \mathrm{cof}(> \mu)$ *and a sequence* $\langle C_\alpha : \alpha \in Z \rangle$
such that for all $\alpha \in Z$*:*

- $C_\alpha \subset \alpha$ *is a club in* α*;*
- *the order-type of* C_α *is* $< \lambda$*;*
- *for all* $\beta \in \lim C_\alpha$*,* $\beta \in S$ *and* $C_\alpha \cap \beta = C_\beta$*.*

Proof of WSTfS: Defining the Partial Square

Proof of WSTfS: Defining the Partial Square

- We have a singular λ with cofinality $\mu > \omega$ such that $\{\delta < \lambda : \Box_\delta \text{ holds}\}$ is stationary. Let $\langle \lambda_i : i < \mu \rangle$ be a club in λ. Then $S := \{i < \mu : \Box_{\lambda_i} \text{ holds}\}$ is stationary in μ.

Proof of WSTfS: Defining the Partial Square

- We have a singular λ with cofinality $\mu > \omega$ such that $\{\delta < \lambda : \square_\delta \text{ holds}\}$ is stationary. Let $\langle \lambda_i : i < \mu \rangle$ be a club in λ. Then $S := \{i < \mu : \square_{\lambda_i} \text{ holds}\}$ is stationary in μ.
- Using the fact that $\prod_{i \in S} \lambda_i^+$ carries a good scale, we can argue that there is a *continuous* good scale $\vec{f}$. Namely, a scale $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:
  - If $\text{cf}\,\alpha < \mu$ and $A \subset \alpha$ is unbounded, then $[i \mapsto \sup_{\beta \in A} f_\beta(i)] =^* f_\alpha$.
  - If $\text{cf}\,\alpha > \mu$, then for all unbounded $A \subset \alpha$, there is some $A' \subset A$ unbounded such that $[i \mapsto \sup_{\beta \in A'} f_\beta(i)] =^* f_\alpha$.

Proof of WSTfS: Defining the Partial Square

- We have a singular λ with cofinality $\mu > \omega$ such that $\{\delta < \lambda : \Box_\delta \text{ holds}\}$ is stationary. Let $\langle \lambda_i : i < \mu \rangle$ be a club in λ. Then $S := \{i < \mu : \Box_{\lambda_i} \text{ holds}\}$ is stationary in μ.
- Using the fact that $\prod_{i \in S} \lambda_i^+$ carries a good scale, we can argue that there is a *continuous* good scale $\vec{f}$. Namely, a scale $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:
  - If $\operatorname{cf} \alpha < \mu$ and $A \subset \alpha$ is unbounded, then $[i \mapsto \sup_{\beta \in A} f_\beta(i)] =^* f_\alpha$.
  - If $\operatorname{cf} \alpha > \mu$, then for all unbounded $A \subset \alpha$, there is some $A' \subset A$ unbounded such that $[i \mapsto \sup_{\beta \in A'} f_\beta(i)] =^* f_\alpha$.
- Let $\mathcal{C}_i = \langle C_\xi^i : \xi < \lambda_i^+ \rangle$ witness $\Box_{\lambda_i}$ for all $i \in S$. For each $\alpha \in \lambda^+ \cap \operatorname{cof}(> \mu)$, let:

$$X_\alpha := \langle \beta < \alpha : \{i < \mu : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \subset S \setminus j, \text{ some } j \rangle.$$

Proof of WSTfS: Defining the Partial Square

- We have a singular λ with cofinality $\mu > \omega$ such that $\{\delta < \lambda : \Box_\delta \text{ holds}\}$ is stationary. Let $\langle \lambda_i : i < \mu \rangle$ be a club in λ. Then $S := \{i < \mu : \Box_{\lambda_i} \text{ holds}\}$ is stationary in μ.
- Using the fact that $\prod_{i \in S} \lambda_i^+$ carries a good scale, we can argue that there is a *continuous* good scale $\vec{f}$. Namely, a scale $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:
  - If $\operatorname{cf} \alpha < \mu$ and $A \subset \alpha$ is unbounded, then $[i \mapsto \sup_{\beta \in A} f_\beta(i)] =^* f_\alpha$.
  - If $\operatorname{cf} \alpha > \mu$, then for all unbounded $A \subset \alpha$, there is some $A' \subset A$ unbounded such that $[i \mapsto \sup_{\beta \in A'} f_\beta(i)] =^* f_\alpha$.
- Let $\mathcal{C}_i = \langle C^i_\xi : \xi < \lambda_i^+ \rangle$ witness $\Box_{\lambda_i}$ for all $i \in S$. For each $\alpha \in \lambda^+ \cap \operatorname{cof}(> \mu)$, let:

$$X_\alpha := \langle \beta < \alpha : \{i < \mu : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \subset S \setminus j, \text{ some } j \rangle.$$

- The X_α's are closed under sequences of length $\neq \mu$.

Proof of WSTfS: Defining the Partial Square

- We have a singular λ with cofinality $\mu > \omega$ such that $\{\delta < \lambda : \square_\delta \text{ holds}\}$ is stationary. Let $\langle \lambda_i : i < \mu \rangle$ be a club in λ. Then $S := \{i < \mu : \square_{\lambda_i} \text{ holds}\}$ is stationary in μ.
- Using the fact that $\prod_{i \in S} \lambda_i^+$ carries a good scale, we can argue that there is a *continuous* good scale $\vec{f}$. Namely, a scale $\langle f_\alpha : \alpha < \lambda^+ \rangle$ such that:
    - If $\operatorname{cf} \alpha < \mu$ and $A \subset \alpha$ is unbounded, then $[i \mapsto \sup_{\beta \in A} f_\beta(i)] =^* f_\alpha$.
    - If $\operatorname{cf} \alpha > \mu$, then for all unbounded $A \subset \alpha$, there is some $A' \subset A$ unbounded such that $[i \mapsto \sup_{\beta \in A'} f_\beta(i)] =^* f_\alpha$.
- Let $\mathcal{C}_i = \langle C^i_\xi : \xi < \lambda_i^+ \rangle$ witness $\square_{\lambda_i}$ for all $i \in S$. For each $\alpha \in \lambda^+ \cap \operatorname{cof}(> \mu)$, let:

$$X_\alpha := \langle \beta < \alpha : \{i < \mu : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \subset S \setminus j, \text{ some } j \rangle.$$

- The X_α's are closed under sequences of length $\neq \mu$.
- C_α will be the closure of X_α in α.

Proof of WSTfS: Verifying the Properties of Partial Square

- $X_\alpha := \langle \beta < \alpha : \{i < \mu : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \subset S \setminus j, \text{ some } j \rangle$

Proof of WSTfS: Verifying the Properties of Partial Square

- $X_\alpha := \langle \beta < \alpha : \{i < \mu : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \subset S \setminus j, \text{ some } j \rangle$
- Coherence of the X_α's follows (with an argument) from closure.

Proof of WSTfS: Verifying the Properties of Partial Square

- $X_\alpha := \langle \beta < \alpha : \{i < \mu : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \subset S \setminus j, \text{ some } j \rangle$
- Coherence of the X_α's follows (with an argument) from closure.
- Unboundedness of the X_α's follows from an interleaving argument: We define $\langle \gamma_n : n < \omega \rangle \subset \alpha$, $\langle g_n : n < \omega \rangle \subset \prod_{i<\mu} \lambda_i^+$, $\langle j_n : n < \omega \rangle \subset \mu$ such that given γ_n, $g_{n+1}(i) = \min C^i_{f_\alpha(i)} \setminus f_{\gamma_n}(i)$ and $g <^* f_{\gamma_{n+1}}$ (using the exact upper bound property). Then if $\gamma^* = \sup_{n<\omega} \gamma_n$ and $j^* = \sup_{n<\omega} j_n$, j^* witnesses that $\gamma^* \in X_\alpha$.

Proof of WSTfS: Verifying the Properties of Partial Square

- $X_\alpha := \langle \beta < \alpha : \{i < \mu : f_\beta(i) \in \lim C^i_{f_\alpha(i)}\} \subset S \setminus j, \text{ some } j\rangle$
- Coherence of the X_α's follows (with an argument) from closure.
- Unboundedness of the X_α's follows from an interleaving argument: We define $\langle \gamma_n : n < \omega\rangle \subset \alpha$, $\langle g_n : n < \omega\rangle \subset \prod_{i<\mu} \lambda_i^+$, $\langle j_n : n < \omega\rangle \subset \mu$ such that given γ_n, $g_{n+1}(i) = \min C^i_{f_\alpha(i)} \setminus f_{\gamma_n}(i)$ and $g <^* f_{\gamma_{n+1}}$ (using the exact upper bound property). Then if $\gamma^* = \sup_{n<\omega} \gamma_n$ and $j^* = \sup_{n<\omega} j_n$, j^* witnesses that $\gamma^* \in X_\alpha$.
- To show that the X_α's have order-type $< \lambda$: Assume without loss of generality that the square sequences $\langle C^i_\xi : i < \mu\rangle$ were defined so that ot $C^i_\xi < \lambda_i$ for all $\xi < \lambda_i^+, i < \mu$. For every $i \in \lim S$, there is some $j < i$ such that ot $C^i_{f_\alpha(i)} < \lambda_j$. So there is a stationary $T \subseteq S$ and some j such that for all $i \in T$, ot $C^i_{f_\alpha(i)} < \lambda_j$. If $\beta \in X_\alpha$, let $g_\beta(i) = \text{ot } C^i_{f_\alpha(i)} \cap f_\beta(i)$ for all i such that $f_\beta(i) \in C^i_{f_\alpha(i)}$ and 0 otherwise. If $\beta, \beta' \in X_\alpha$ and $\beta < \beta'$, then g_β and $g_{\beta'}$ are distinct because f'_β eventually dominates f_β. Furthermore, $\{g_\beta : \beta \in X_\alpha\}$ has size $\lambda_j^{\text{cf}\,\lambda} < \lambda$ since we assumed λ is a strong limit, which shows that X_β has size less than λ.

Conjectures Not Necessarily Believed by Collaborators

Conjectures Not Necessarily Believed by Collaborators

Conjecture

Silver's Theorem for Square is false. Rather, it is consistent up to large cardinals that $\square_{\aleph_\delta}$ holds for all $\delta < \omega_1$, while $\square_{\aleph_{\omega_1}}$ fails.

Conjectures Not Necessarily Believed by Collaborators

Conjecture

Silver's Theorem for Square is false. Rather, it is consistent up to large cardinals that $\square_{\aleph_\delta}$ holds for all $\delta < \omega_1$, while $\square_{\aleph_{\omega_1}}$ fails.

Conjecture

Some strong-ish form of Silver's Theorem for Weak Square is true. If λ is a singular strong limit of uncountable cofinality and $\square^*_\delta$ holds for all $\delta < \lambda$, then $\square^*_\lambda$ holds as well.

Děkuji!